Cambridge Assessment International Education

Cambridge International AS & A Level

FURTHER MATHEMATICS **9231/22**

Paper 2 Further Pure Mathematics 2 **October/November 2023**

MARK SCHEME

Maximum Mark: 75

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Due to a series-specific issue during the live exam series, all candidates were awarded full marks for question 6. This published mark scheme for these questions was created alongside the question paper but has not been used by examiners.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge International will not enter into discussions about these mark schemes.

Cambridge International is publishing the mark schemes for the October/November 2023 series for most Cambridge IGCSE, Cambridge International A and AS Level components, and some Cambridge O Level components.

Generic Marking Principles

These general marking principles must be applied by all examiners when marking candidate answers. They should be applied alongside the specific content of the mark scheme or generic level descriptors for a question. Each question paper and mark scheme will also comply with these marking principles.

GENERIC MARKING PRINCIPLE 1: Marks must be awarded in line with: • the specific content of the mark scheme or the generic level descriptors for the question • the specific skills defined in the mark scheme or in the generic level descriptors for the question • the standard of response required by a candidate as exemplified by the standardisation scripts.
GENERIC MARKING PRINCIPLE 2: Marks awarded are always **whole marks** (not half marks, or other fractions).
GENERIC MARKING PRINCIPLE 3: Marks must be awarded **positively**: • marks are awarded for correct/valid answers, as defined in the mark scheme. However, credit is given for valid answers which go beyond the scope of the syllabus and mark scheme, referring to your Team Leader as appropriate • marks are awarded when candidates clearly demonstrate what they know and can do • marks are not deducted for errors • marks are not deducted for omissions • answers should only be judged on the quality of spelling, punctuation and grammar when these features are specifically assessed by the question as indicated by the mark scheme. The meaning, however, should be unambiguous.
GENERIC MARKING PRINCIPLE 4: Rules must be applied consistently, e.g. in situations where candidates have not followed instructions or in the application of generic level descriptors.
GENERIC MARKING PRINCIPLE 5: Marks should be awarded using the full range of marks defined in the mark scheme for the question (however; the use of the full mark range may be limited according to the quality of the candidate responses seen).
GENERIC MARKING PRINCIPLE 6: Marks awarded are based solely on the requirements as defined in the mark scheme. Marks should not be awarded with grade thresholds or grade descriptors in mind.

Mathematics-Specific Marking Principles

1 Unless a particular method has been specified in the question, full marks may be awarded for any correct method. However, if a calculation is required then no marks will be awarded for a scale drawing.

2 Unless specified in the question, non-integer answers may be given as fractions, decimals or in standard form. Ignore superfluous zeros, provided that the degree of accuracy is not affected.

3 Allow alternative conventions for notation if used consistently throughout the paper, e.g. commas being used as decimal points.

4 Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored (isw).

5 Where a candidate has misread a number or sign in the question and used that value consistently throughout, provided that number does not alter the difficulty or the method required, award all marks earned and deduct just 1 A or B mark for the misread.

6 Recovery within working is allowed, e.g. a notation error in the working where the following line of working makes the candidate's intent clear.

Mark Scheme Notes

The following notes are intended to aid interpretation of mark schemes in general, but individual mark schemes may include marks awarded for specific reasons outside the scope of these notes.

Types of mark

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

DM or **DB** When a part of a question has two or more 'method' steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly, when there are several B marks allocated. The notation DM or DB is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

FT Implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only.

- A or B marks are given for correct work only (not for results obtained from incorrect working) unless follow through is allowed (see abbreviation FT above).
- For a numerical answer, allow the A or B mark if the answer is correct to 3 significant figures or would be correct to 3 significant figures if rounded (1 decimal place for angles in degrees).
- The total number of marks available for each question is shown at the bottom of the Marks column.
- Wrong or missing units in an answer should not result in loss of marks unless the guidance indicates otherwise.
- Square brackets [ ] around text or numbers show extra information not needed for the mark to be awarded.

Abbreviations

AEF/OE	Any Equivalent Form (of answer is equally acceptable) / Or Equivalent
AG	Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)
CAO	Correct Answer Only (emphasising that no 'follow through' from a previous error is allowed)
CWO	Correct Working Only
ISW	Ignore Subsequent Working
SOI	Seen Or Implied
SC	Special Case (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)
WWW	Without Wrong Working
AWRT	Answer Which Rounds To

Question	Answer	Marks	Guidance
1	**EITHER** Solution 1 $\ln(x+2)=\ln\left(2\left(1+\frac{1}{2}x\right)\right)=\ln 2+\ln\left(1+\frac{1}{2}x\right)$	**M1**	Changes $\ln(x+2)$ or $\ln(x^2+5)$ so that the formula given in the list of formulae (MF19) can be applied.
	$\ln(x+2)=\ln 2+\frac{1}{2}x-\frac{1}{8}x^2+\ldots$	**A1**	
	$\ln(x^2+5)=\ln\left(5\left(1+\frac{1}{5}x^2\right)\right)=\ln 5+\frac{1}{5}x^2+\ldots$	**A1**	
	$\left(\ln 2+\frac{1}{2}x-\frac{1}{8}x^2+\ldots\right)+\left(\ln 5+\frac{1}{5}x^2+\ldots\right)$	**M1**	Sums power series
	OR Solution 2 $f'(x)=(x+2)^{-1}+2x(x^2+5)^{-1}$	**(M1 A1)**	Finds first derivative.
	$f''(x)=-(x+2)^{-2}-(2x)^2(x^2+5)^{-2}+2(x^2+5)^{-1}$	**(A1)**	Finds second derivative.
	$f(0)=\ln 10 \quad f'(0)=\frac{1}{2} \quad f''(0)=\frac{3}{20}$	**(M1)**	Evaluates derivatives at zero.
	$\ln(x+2)+\ln(x^2+5)=\ln 10+\frac{1}{2}x+\frac{3}{40}x^2$	**A1**	Accept $\ln 10$ written as $\ln 2+\ln 5$ but do not accept decimals. WWW.
		5	

Question	Answer	Marks	Guidance
2(a)	$\frac{dy}{dt} = e^t(t+1)$, $\frac{dx}{dt} = -t^{-2}$	**B1**	
	$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = -t^2 e^t(t+1) = -e^t(t^3+t^2)$	**M1 A1**	Uses chain rule, AG.
		3	
2(b)	$\frac{d}{dt}\left(\frac{dy}{dx}\right) = -e^t(3t^2+2t) - e^t(t^3+t^2) = -e^t(t^3+4t^2+2t)$	**M1 A1**	Applies product rule.
	$\frac{d^2y}{dx^2} = t^3 e^t(t^2+4t+2)$	**M1 A1**	Uses chain rule, multiplies by *their* $\frac{dt}{dx}$ for M1.
	Alternative method for question 2(b)		
	$\frac{d^2y}{dx^2} = -e^t\left(3t^2\frac{dt}{dx} + 2t\frac{dt}{dx}\right) - e^t(t^3+t^2)\frac{dt}{dx}$	**(M1 A1)**	Differentiates $-e^t(t^3+t^2)$ implicitly.
	$\frac{d^2y}{dx^2} = e^t(-4t^2-2t-t^3)\frac{dt}{dx} = t^3e^t(t^2+4t+2)$	**(M1 A1)**	Substitutes $\frac{dt}{dx} = -t^2$.
		4	

Question	Answer	Marks	Guidance
3(a)	$\cos 5\theta = \mathrm{Re}(\cos\theta + \mathrm{i}\sin\theta)^5 = \cos^5\theta - 10\sin^2\theta\cos^3\theta + 5\sin^4\theta\cos\theta$	M1 A1	Expands and takes real part. Accept RHS to LHS using $2\cos\theta = z + \frac{1}{z}$.
	$= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1 - 2\cos^2\theta + \cos^4\theta)$	M1	Applies $\sin^2\theta = 1 - \cos^2\theta$.
	$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$	A1	AG
		4	
3(b)	$x = \cos\theta, \quad \cos 5\theta = \frac{1}{2}\sqrt{2}$	M1	Applies identify given in **(b)**.
	$5\theta = \pm\frac{1}{4}\pi + 2k\pi$	M1	Solves $\cos 5\theta = \frac{1}{2}\sqrt{2}$
	$\cos\left(\frac{1}{20}\pi\right)$	A1	Gives one correct solution. Accept $q = \frac{1}{20}$.
	$\cos(\frac{9}{20}\pi), \cos(\frac{17}{20}\pi), \cos(\frac{25}{20}\pi), \cos(\frac{33}{20}\pi)$	A1	Gives other solutions. OE. A0 for repeated roots.
		4	

Question	Answer	Marks	Guidance
4	$e^{\int 3\,dx} = e^{3x}$	**M1 A1**	Finds integrating factor.
	$\frac{d}{dx}\left(ye^{3x}\right) = e^{3x}\sin x$	**M1**	Correct form on LHS and attempt to integrate RHS.
	EITHER $\int e^{3x}\sin x\,dx = -e^{3x}\cos x + 3\int e^{3x}\cos x\,dx$ **OR** $\int e^{3x}\sin x\,dx = \frac{1}{3}e^{3x}\sin x - \frac{1}{3}\int e^{3x}\cos x\,dx$	**M1 A1**	Integrates by parts once or uses $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$.
	EITHER $\int e^{3x}\sin x\,dx = -e^{3x}\cos x + 3\left(e^{3x}\sin x - 3\int e^{3x}\sin x\,dx\right)$ **OR** $\int e^{3x}\sin x\,dx = \frac{1}{3}e^{3x}\sin x - \frac{1}{3}\left(\frac{1}{3}e^{3x}\cos x + \frac{1}{3}\int e^{3x}\sin x\,dx\right)$	**M1**	Integrates by parts again or substitutes $\frac{e^{ix} - e^{-ix}}{2i} = \sin x$ and $\frac{e^{ix} + e^{-ix}}{2} = \cos x$.
	$ye^{3x} = \frac{1}{10}e^{3x}\left(3\sin x - \cos x\right) + C$	**A1**	Must not see i.
	$1 = -\frac{1}{10} + C$	**M1**	Finds C. Substitutes into their expression (must be integrated).
	$y = \frac{3}{10}\sin x - \frac{1}{10}\cos x + \frac{11}{10}e^{-3x}$	**A1**	Divides through by coefficient of y.
		9	

Question	Answer	Marks	Guidance
5(a)	$\frac{dy}{dx} = -2x\operatorname{sech}^2 x \tanh x + \operatorname{sech}^2 x$ $\left[= 1 - \tanh^2 x - 2x\tanh x + 2x\tanh^3 x\right]$	M1 A1	Differentiating using product rule and chain rule for M1.
	$\operatorname{sech}^2 x \neq 0 \Rightarrow 2x\tanh x - 1 = 0$	A1	AG.
	$2(0.7)\tanh(0.7) - 1 = -0.154 < 0$ $2(0.8)\tanh(0.8) - 1 = 0.062 > 0$	B1	Shows sign change. Must write down values correct to at least 1dp for B1.
		4	
5(b)	$\sum_{r=2}^{n} r\operatorname{sech}^2 r < \int_1^n x\operatorname{sech}^2 x \, dx$	M1 A1	Compares sum with integral. Consistent limits for M1.
	$\int_1^n x\operatorname{sech}^2 x \, dx = \left[x\tanh x\right]_1^n - \int_1^n \tanh x \, dx$	M1 A1	Integrates by parts.
	$= \left[x\tanh x + \ln\operatorname{sech} x\right]_1^n$	A1	
	$= n\tanh n + \ln\operatorname{sech} n - (\tanh 1 + \ln\operatorname{sech} 1)$	A1	AG. Must have gained all previous marks.
		6	

Question	Answer	Marks	Guidance
6(a)	1, 2, -1	B1	
		1	

Question	Answer	Marks	Guidance
6(b)	$\mathbf{P}^3 - 2\mathbf{P}^2 - \mathbf{P} + 2\mathbf{I} = 0$	B1	States that **P** satisfies its characteristic equation.
	$2\mathbf{P}^{-1} = \mathbf{I} + 2\mathbf{P} - \mathbf{P}^2$	M1	Multiplies through by $\mathbf{P}^{-1}$.
	$\mathbf{P}^2 = \begin{pmatrix} 1 & -3 & -1 \\ 0 & 4 & 1 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \mathbf{P}^{-1} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -1 \end{pmatrix}$	M1 A1	
		4	
6(c)	$\mathbf{D} = \begin{pmatrix} \frac{1}{a} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$	B1	
	$\mathbf{A}^{-1} = \mathbf{P}\begin{pmatrix} \frac{1}{a} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}\mathbf{P}^{-1}$	M1	Applies $\mathbf{A}^{-1} = \mathbf{PDP}^{-1}$.
	$= \begin{pmatrix} \frac{1}{a} & -2 & \frac{1}{2} \\ 0 & 4 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -1 \end{pmatrix}$	M1 A1	Multiplies two adjacent matrices.
	$\begin{pmatrix} \frac{1}{a} & \frac{1-2a}{2a} & \frac{3-3a}{2a} \\ 0 & 2 & \frac{3}{2} \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$	A1	
		5	

Question	Answer	Marks	Guidance
7(a)	$\cosh 2A = \frac{1}{2}\left(e^{2A} + e^{-2A}\right) \quad \sinh A = \frac{1}{2}\left(e^{A} - e^{-A}\right)$	**B1**	
	$2\sinh^2 A = \frac{1}{4}\left(e^{A} - e^{-A}\right)^2 = \frac{1}{2}\left(e^{2A} - 2 + e^{-2A}\right) = \cosh 2A - 1$	**M1 A1**	Expands, AG. A0 for mixing variables e.g. $\sinh A = \frac{1}{2}\left(e^{x} - e^{-x}\right)$.
		3	

Question	Answer	Marks	Guidance
7(b)	$S=\int_0^{\frac{2}{3}} 2\pi y\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x=2\pi\int_0^{\frac{2}{3}} x^2\sqrt{1+4x^2}\,\mathrm{d}x$	M1 A1	Correct formula with correct limits. Correct limits for M1. (Limits may be recovered.)
	$S=\pi\int_0^{\sinh^{-1}\frac{4}{3}} \frac{1}{4}\sinh^2 u\sqrt{1+\sinh^2 u}\cosh u\,\mathrm{d}u$	M1	Applies given substitution to their expression with correct limits. (Limits may be recovered.)
	$=\frac{1}{4}\pi\int_0^{\sinh^{-1}\frac{4}{3}} \sinh^2 u\cosh^2 u\,\mathrm{d}u$	A1	Must be simplified.
	$=\frac{1}{16}\pi\int_0^{\sinh^{-1}\frac{4}{3}} \sinh^2 2u\,\mathrm{d}u$	M1	Applies $\sinh 2u=2\sinh u\cosh u$. May use double angle formulae for cosh and sinh instead.
	$=\frac{1}{32}\pi\int_0^{\sinh^{-1}\frac{4}{3}} \cosh 4u-1\,\mathrm{d}u$	M1	Applies $\sinh^2 A=\frac{1}{2}(\cosh 2A-1)$. S must have the form $a\int\sinh^2 u\cosh^2 u\,\mathrm{d}u$.
	$=\frac{1}{32}\pi\left[\frac{1}{4}\sinh 4u-u\right]_0^{\sinh^{-1}\frac{4}{3}}$	A1	
	$\sinh^{-1}\frac{4}{3}=\ln 3$	B1	
	$S=\frac{1}{32}\pi\left(\frac{1}{8}\left(\mathrm{e}^{\ln 81}-\mathrm{e}^{-\ln 81}\right)-\ln 3\right)=\frac{1}{32}\pi\left(\frac{1}{8}\left(81-\frac{1}{81}\right)-\ln 3\right)=\frac{1}{32}\pi\left(\frac{820}{81}-\ln 3\right)$	A1	AG.
		9	

Question	Answer	Marks	Guidance
8(a)	$\frac{dv}{dx} = 4y^3\frac{dy}{dx}$	B1	$v = y^4$
	$\frac{d^2v}{dx^2} = 4y^3\frac{d^2y}{dx^2} + 12y^2\left(\frac{dy}{dx}\right)^2$	B1	
	$\frac{d^2v}{dx^2} + \frac{dv}{dx} + 4v = 4y^3\frac{d^2y}{dx^2} + 12y^2\left(\frac{dy}{dx}\right)^2 + 4y^3\frac{dy}{dx} + 4y^4$	M1	Uses substitution to find *v-x* equation, AG.
	$= 4\left(y^3\frac{d^2y}{dx^2} + 3y^2\left(\frac{dy}{dx}\right)^2 + y^3\frac{dy}{dx} + y^4\right) = 4e^{-2x}$	A1	AG.
	Alternative method for question 8(a)		
	$\frac{dy}{dx} = \frac{1}{4y^3}\frac{dv}{dx} = \frac{1}{4v^{\frac{3}{4}}}\frac{dv}{dx}$	(B1)	$y = v^{\frac{1}{4}}$
	$\frac{d^2y}{dx^2} = \frac{1}{4v^{\frac{3}{4}}}\frac{d^2v}{dx^2} - \frac{3}{16v^{\frac{7}{4}}}\left(\frac{dv}{dx}\right)^2$	(B1)	
	$v^{\frac{3}{4}}\left(\frac{1}{4v^{\frac{3}{4}}}\frac{d^2v}{dx^2} - \frac{3}{16v^{\frac{7}{4}}}\left(\frac{dv}{dx}\right)^2\right) + 3v^{\frac{1}{2}}\left(\frac{1}{4v^{\frac{3}{4}}}\frac{dv}{dx}\right)^2 + v^{\frac{3}{4}}\left(\frac{1}{4v^{\frac{3}{4}}}\frac{dv}{dx}\right) + v = e^{-2x}$	(M1)	Uses substitution to find *v-x* equation, AG.
	$\frac{1}{4}\frac{d^2v}{dx^2} - \frac{3}{16v}\left(\frac{dv}{dx}\right)^2 + \frac{3}{16v}\left(\frac{dv}{dx}\right)^2 + \frac{1}{4}\frac{dv}{dx} + v = e^{-2x}$	(A1)	
		4	

Question	Answer	Marks	Guidance
8(b)	$m^2 + m + 4 = 0$	M1	Auxiliary equation.
	$[v =]\mathrm{e}^{-\frac{1}{2}x}\left(A\cos\frac{\sqrt{15}}{2}x + B\sin\frac{\sqrt{15}}{2}x\right)$	A1	Complimentary function. Allow '$v =$' missing.
	$v = k\mathrm{e}^{-2x} \Rightarrow v' = -2k\mathrm{e}^{-2x} \Rightarrow v'' = 4k\mathrm{e}^{-2x}$	B1	Particular integral and its derivatives.
	$4k\mathrm{e}^{-2x} - 2k\mathrm{e}^{-2x} + 4k\mathrm{e}^{-2x} = 4\mathrm{e}^{-2x}$	M1	Substitutes and equates coefficients.
	$k = \frac{2}{3}$	A1	
	$v = y^4 = \mathrm{e}^{-\frac{1}{2}x}\left(A\cos\frac{\sqrt{15}}{2}x + B\sin\frac{\sqrt{15}}{2}x\right) + \frac{2}{3}\mathrm{e}^{-2x}$	A1	Substitutes for y and find v in terms of x. Must not see i.
	$v' = 4y^3\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{-\frac{1}{2}x}\left(-\frac{\sqrt{15}}{2}A\sin\frac{\sqrt{15}}{2}x + \frac{\sqrt{15}}{2}B\cos\frac{\sqrt{15}}{2}x\right) - \frac{1}{2}\mathrm{e}^{-\frac{1}{2}x}\left(A\cos\frac{\sqrt{15}}{2}x + B\sin\frac{\sqrt{15}}{2}x\right) - \frac{4}{3}\mathrm{e}^{-2x}$	B1	
	$1 = A + \frac{2}{3} \qquad -\frac{3}{2} = \frac{\sqrt{15}}{2}B - \frac{1}{2}A - \frac{4}{3} \qquad \Rightarrow A = \frac{1}{3}, \quad B = 0$	M1 A1	Substitutes initial conditions and forms simultaneous equations. Must have used the product rule when differentiating *their* v for M1.
	$y = \left(\frac{1}{3}\mathrm{e}^{-\frac{1}{2}x}\cos\frac{\sqrt{15}}{2}x + \frac{2}{3}\mathrm{e}^{-2x}\right)^{\frac{1}{4}}$	A1	Accept $y = \pm\left(\frac{1}{3}\mathrm{e}^{-\frac{1}{2}x}\cos\frac{\sqrt{15}}{2}x + \frac{2}{3}\mathrm{e}^{-2x}\right)^{\frac{1}{4}}$
		10	